

A SAMPLING INEQUALITY FOR FRACTIONAL ORDER SOBOLEV SEMI-NORMS USING ARBITRARY ORDER DATA

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ABSTRACT. To improve convergence results obtained using a framework for unsymmetric meshless methods due to Schaback (Preprint Göttingen 2006), we extend, in two directions, the Sobolev bound due to Arcangéli et al. (Numer Math 107, 181-211, 2007), which itself extends two others due to Wendland and Rieger (Numer Math 101, 643-662, 2005) and Madych (J. Approx Theory 142, 116-128, 2006). The first is to incorporate discrete samples of arbitrary order derivatives into the bound, which are used to obtain higher order convergence in higher order Sobolev norms. The second is to optimally bound fractional order Sobolev semi-norms, which are used to obtain more optimal convergence rates when solving problems requiring fractional order Sobolev spaces, notably inhomogeneous boundary value problems.

1. INTRODUCTION

Over the past few years, increasingly general bounds of Sobolev semi-norms, in terms of discrete samples, have appeared. Such bounds are often called *sampling inequalities*. A rather general sampling inequality was established by Arcangéli et al. [1], and is stated as Theorem 1.1, with notation given in Section 1.1.

Theorem 1.1. [1, Theorem 4.1] *Let Ω be a Lipschitz domain in \mathbb{R}^n , so that the domain Ω satisfies the cone property [1, Page 185] with radius $\rho > 0$ and angle $\theta \in (0, \pi/2]$. Furthermore, let $p, q, \varkappa \in [1, \infty]$ and let r be a real number such that $r \geq n$, if $p = 1$, $r > n/p$ if $1 < p < \infty$, or $r \in \mathbb{N}^*$, if $p = \infty$. Let $l_0 = r - n(1/p - 1/q)_+$ and $\gamma = \max\{p, q, \varkappa\}$. Then, there exist two positive constants \mathfrak{d}_r (dependent on θ, ρ, n and r) and C (dependent on Ω, n, r, p, q and \varkappa) satisfying the following property: for any set $A \subset \overline{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and $r = n$) such that $d = \delta(A, \overline{\Omega}) \leq \mathfrak{d}_r$ (c.f. (2)) for any $u \in W^{r,p}(\Omega)$ and for any real number l satisfying $l = 0, \dots, l_{\max}$, we have*

$$(1) \quad |u|_{l,q,\Omega} \leq C \left(d^{r-l-n(1/p-1/q)_+} |u|_{r,p,\Omega} + d^{n/\gamma-l} \|u\|_{\varkappa} \right).$$

where $l_{\max} := [l_0] - 1$, unless the following additional conditions hold, in which case $l_{\max} := l_0$: $r \in \mathbb{N}^*$ and either (i) $p < q < \infty$ and $l_0 \in \mathbb{N}$, (ii) $(p, q) = (1, \infty)$, or (iii) $p \geq q$.

This sampling inequality generalizes those of Madych [6] and Wendland and Rieger [15], by greatly extending the range of parameters r, p, l , and \varkappa . While Theorem 1.1 applies to functions with finite smoothness, an analogous bound for functions with infinite smoothness has been provided by Rieger and Zwicknagl [10] which achieves exponential factors.

Arcangéli, et al. [1] used Theorem 1.1 to derive error bounds for interpolating and smoothing (m, s) -splines, an application which we do not consider. Instead we

are interested in another major application of these Sobolev estimates: Schaback's framework for unsymmetric meshless methods for operator equations [12], see also the earlier version [13]. A sampling inequality is necessary for unsymmetric meshless methods, such as Schaback's modification of Kansa's method [12, 13], which involve an overdetermined system of equations in general. In an attempt to improve the order of convergence obtained using Schaback's framework, we extend the bound of Arcangéli, et al. in two ways.

Our first extension is to loosen the restriction $l \in \mathbb{N}$ to allow for fractional order Sobolev norms on the left hand side of the sampling inequality. In the context of Schaback's framework, this will result in more optimal convergence results in terms of both the test and trial discretization parameters. Otherwise, the test discretization would require a higher rate of refinement.

Our second extension is to incorporate discrete samples of arbitrary order derivatives into the bound. The reason for this is that (1) has a factor d^{-l} in its second term which is insufficient for achieving a *uniformly stable test discretization* for higher order Sobolev norms in Schaback's framework. With this modification to incorporate samples of higher order derivatives we will be able to come closer to achieving such a test discretization, resulting in higher order convergence results. This introduces a new parameter μ , for which previous sampling inequalities coincide with the choice $\mu = 0$.

1.1. Notation. We employ the notation of Arcangéli, et al. [1, Section 2].

As in [1, Section 4], we assume throughout this paper that Ω is a bounded domain in \mathbb{R}^n with a Lipschitz-continuous boundary, so that the domain Ω satisfies the cone property [1, Page 185] with radius $\rho > 0$ and angle $\theta \in (0, \pi/2]$. For a given finite subset A of $\overline{\Omega}$, the *fill distance* is defined as

$$(2) \quad \delta(A, \overline{\Omega}) = \sup_{x \in \Omega} \min_{a \in A} |x - a|.$$

The following restates a portion of their notation. For all $r \in [0, \infty]$ and $p \in [1, \infty]$, the Sobolev norm is denoted by $\|\cdot\|_{r,p,\Omega}$, while the Sobolev semi-norm is denoted by $|\cdot|_{r,p,\Omega}$. The set $\mathbb{N}^* = \{1, 2, 3, \dots\}$, while $\mathbb{N} = \{0, 1, 2, \dots\}$. The space of polynomials over \mathbb{R}^n with degree less than or equal to k is denoted by P_k .

We make the following additions to their notation. Let $H^r(\Omega) := W^{r,2}(\Omega)$ and

$$\tilde{W}^{r,q}(\Omega) := \left\{ v \in W^{r,q}(\Omega) : \int_{\Omega} v = 0 \right\}.$$

Given a function $v \in W^{1,q}(\Omega)$, the vector-valued function consisting of its partial derivatives is denoted by Dv . The surface area of the n -dimensional ball is denoted by $|S^{n-1}|$. In Section 2, a generic constant C appears in many proofs, whose particular value may change, but with the parameters on which it depends either indicated in parentheses or stated explicitly in the exposition. We will often substitute dependencies with others, possibly taking the maximum or minimum value, as required by its application, of the constant over a finite range of values. In Section 3, only the dependence of constants on the discretization parameters r and s is explicitly stated since we regard the spaces and mappings in that section as fixed.

2. EXTENSION OF THE SOBOLEV BOUND

2.1. Fractional Order Sobolev Spaces. This section concerns fractional order Sobolev norms and the results of this section will be used to generalize [1, Proposition 3.4] to Proposition 2.7. Lemmas 2.2 and 2.3 each require an extension operator which satisfies (4) for zero-average functions over a ball. This property is not provided by standard extension operators since they involve a domain-dependent constant and the full Sobolev norm in the bound, rather than a domain-independent constant and the Sobolev semi-norm.

Lemma 2.1. *If $q \in [1, \infty]$, $r > 0$, and $x_0 \in \mathbb{R}^n$ then there exists a linear, continuous operator*

$$E : \tilde{W}^{1,q}(B(x_0, r)) \rightarrow W^{1,q}(\mathbb{R}^n)$$

such that for all $v \in \tilde{W}^{1,q}(B(x_0, r))$

$$(3) \quad Ev = v \text{ a.e. in } B(x_0, r),$$

and

$$(4) \quad |Ev|_{1,q,\mathbb{R}^n} \leq C(n, q) |v|_{1,q,B(x_0,r)}.$$

If in addition $v \in C^1(\overline{B(x_0, r)})$ then $Ev \in C^1(\mathbb{R}^n)$.

Proof. Let $v \in \tilde{W}^{1,q}(B(x_0, r))$ and $\hat{v} := v \circ F$ where $F : \hat{x} \rightarrow r\hat{x} + x_0$. From a change of variables it follows that $\hat{v} \in W^{1,q}(B(0, 1))$ with semi-norm

$$|\hat{v}|_{1,q,B(0,1)} = r^{1-n/q} |v|_{1,q,B(x_0,r)}.$$

From [5, Section 5.4, Theorem 1], there exists a linear, continuous extension operator

$$\hat{E} : W^{1,q}(B(0, 1)) \rightarrow W^{1,q}(\mathbb{R}^n)$$

such that for each $v \in W^{1,q}(B(0, 1))$

$$\hat{E}\hat{v} = \hat{v} \text{ a.e. in } B(0, 1),$$

and

$$\|\hat{E}\hat{v}\|_{1,q,\mathbb{R}^n} \leq C(n, q) \|\hat{v}\|_{1,q,B(0,1)},$$

where the dependence on n is through $B(0, 1)$. That

$$\|\hat{v}\|_{1,q,B(0,1)} \leq C(n, q) |\hat{v}|_{1,q,B(0,1)},$$

follows from specializing a Poincaré inequality given in [5, Section 5.8, Theorem 2] to the unit ball and that

$$(5) \quad \int_{B(0,1)} \hat{v} = r^{-n} \int_{B(x_0,r)} v = 0.$$

Let

$$(6) \quad Ev := (\hat{E}\hat{v}) \circ F^{-1}$$

so that the result (3) holds. From another change of variables, it follows that

$$|Ev|_{1,q,\mathbb{R}^n} = r^{n/q-1} |\hat{E}\hat{v}|_{1,q,\mathbb{R}^n}.$$

The result (4) follows by combining the preceding relations. Finally, from the proof of [5, Section 5.4, Theorem 1] it follows that if $\hat{v} \in C^1(\overline{B(0,1)})$, then $\hat{E}\hat{v} \in C^1(\mathbb{R}^n)$, so that if $v \in C^1(\overline{B(x_0, r)})$ then $Ev \in C^1(\mathbb{R}^n)$. \square

Based on this extension, we obtain Lemma 2.2, which is similar to a result used by Bourgain et al. [2, Eq. 2], but uses a domain-independent constant and semi-norm in the bound.

Lemma 2.2. *If $q \in [1, \infty)$, $h \in \mathbb{R}^n$, and $v \in \tilde{W}^{1,q}(B(x_0, r))$ then*

$$\left(\int_{B(x_0, r)} |Ev(x+h) - Ev(x)|^q dx \right)^{1/q} \leq C(n, q) |h| |v|_{1,q,B(x_0, r)}$$

Proof. First suppose that v is in the subset

$$(7) \quad C^1(\overline{B(x_0, r)}) \cap \tilde{W}^{1,q}(B(x_0, r))$$

which is dense in $\tilde{W}^{1,q}(B(x_0, r))$. From Lemma 2.1 it follows that

$$\begin{aligned} \int_{B(x_0, r)} |Ev(x+h) - Ev(x)|^q dx &\leq \int_{\mathbb{R}^n} |Ev(x+h) - Ev(x)|^q dx \\ &= \int_{\mathbb{R}^n} \left| \int_0^1 \frac{d}{dt} Ev(x+th) dt \right|^q dx \\ &\leq \int_{\mathbb{R}^n} \int_0^1 \left| \frac{d}{dt} Ev(x+th) \right|^q dt dx \\ &= \int_{\mathbb{R}^n} \int_0^1 |DEv(x+th) \cdot h|^q dt dx \\ &\leq |h|^q \int_{\mathbb{R}^n} \int_0^1 |DEv(x+th)|^q dt dx \\ &= |h|^q \int_0^1 \int_{\mathbb{R}^n} |DEv(x+th)|^q dx dt \\ &\leq |h|^q \int_0^1 |Ev(\cdot + th)|_{1,q,\mathbb{R}^n}^q dt \\ &= |h|^q |Ev|_{1,q,\mathbb{R}^n}^q \\ &\leq C(n, q) |h|^q |v|_{1,q,B(x_0, r)}^q \end{aligned}$$

From this, the result follows for all functions in $\tilde{W}^{1,q}(B(x_0, r))$ via a standard density argument. \square

Lemma 2.3. *If $x, x+h \in B(x_0, r)$, and $v \in \tilde{W}^{1,\infty}(B(x_0, r))$ then*

$$|v(x+h) - v(x)| \leq C(n) |h| |v|_{1,\infty,B(x_0, r)}$$

Proof. In the proof of [5, Section 5.8.2, Theorem 4] it is shown that v is a Lipschitz function with constant $|Ev|_{1,\infty,\mathbb{R}^n}$, where the extension operator constructed in [5, Section 5.4, Theorem 1] is used. However, the extension operator from Lemma 2.1 could be substituted so that the result then follows by (4). \square

In the bound provided by Lemma 2.4 the factor $r^{1-\epsilon}$ will be the key to generalizing sampling inequalities to optimally bound fractional order semi-norms.

Proposition 2.4. *If $q \in [1, \infty]$, $\epsilon \in (0, 1)$, and $v \in W^{1,q}(B(x_0, r))$ then*

$$(8) \quad |v|_{\epsilon,q,B(x_0,r)} \leq C(n, q) (1 - \epsilon)^{-1/q} r^{1-\epsilon} |v|_{1,q,B(x_0,r)}.$$

Proof. Since the semi-norms that appear in (8) are invariant with respect to a shift in value of v by a constant, it suffices to only consider $v \in \tilde{W}^{1,q}(B(x_0, r))$.

Case $q \in [1, \infty)$: Let $y \in B(x_0, r)$ and

$$B(x_0, r) - y := \{x - y : x \in B(x_0, r)\},$$

so that $B(x_0, r) - y \subseteq B(0, 2r)$.

$$\begin{aligned} |v|_{\epsilon,q,B(x_0,r)}^q &= \int_{B(x_0,r)} \int_{B(x_0,r)} \frac{|v(x) - v(y)|^q}{|x - y|^{n+\epsilon q}} dx dy \\ &= \int_{B(x_0,r)} \int_{B(x_0,r)-y} \frac{|v(y+h) - v(y)|^q}{|h|^{n+\epsilon q}} dh dy \\ &\leq \int_{B(x_0,r)} \int_{B(0,2r)} \frac{|Ev(y+h) - Ev(y)|^q}{|h|^{n+\epsilon q}} dh dy \\ &= \int_{B(0,2r)} \frac{\int_{B(x_0,r)} |Ev(y+h) - Ev(y)|^q dy}{|h|^{n+\epsilon q}} dh \\ &\leq C(n, q) \left(\int_{B(0,2r)} \frac{|h|^q}{|h|^{n+\epsilon q}} dh \right) |v|_{1,q,B(x_0,r)}^q \\ &= C(n, q) |S^{n-1}| \left(\int_0^{2r} \frac{\rho^q}{\rho^{n+\epsilon q}} \rho^{n-1} d\rho \right) |v|_{1,q,B(x_0,r)}^q \\ &\leq \frac{C(n, q)}{(1 - \epsilon)q} |S^{n-1}| (2r)^{(1-\epsilon)q} |v|_{1,q,B(x_0,r)}^q \\ &\leq \frac{C(n, q)}{(1 - \epsilon)q} |S^{n-1}| 2^q r^{(1-\epsilon)q} |v|_{1,q,B(x_0,r)}^q. \end{aligned}$$

Case $q = \infty$:

$$\begin{aligned} |v|_{\epsilon,q,B(x_0,r)} &= \operatorname{ess\,sup}_{x,y \in B(x_0,r), x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\epsilon} \\ &\leq C(n, q) (2r)^{1-\epsilon} |v|_{1,q,B(x_0,r)} \\ &\leq 2C(n, q) r^{1-\epsilon} |v|_{1,q,B(x_0,r)}. \end{aligned}$$

□

Remark 2.5. *The explicit constant $(1 - \epsilon)^{-1/q}$, which blows up as ϵ increases towards one, is a manifestation of the “defect” of intrinsic fractional order Sobolev semi-norms studied by Bourgain et al. [2].*

Corollary 2.6. *If $q \in [1, \infty]$, $l \in [0, \infty]$, and $v \in W^{[l],q}(B(x_0, r))$ then*

$$(9) \quad |v|_{l,q,B(x_0,r)} \leq C(n, q, [l]) K([l] - l, q) r^{[l]-l} |v|_{[l],q,B(x_0,r)},$$

where

$$(10) \quad K([l] - l, q) := \begin{cases} 1 & \text{for } l \in \mathbb{N} \text{ or } q = \infty \\ ([l] - l)^{-1/q} & \text{for } l \notin \mathbb{N} \text{ and } q < \infty \end{cases}.$$

2.2. An Auxiliary Result. The following result applies Corollary 2.6 to generalize [1, Proposition 3.4].

Proposition 2.7. *Let $p, q, \varkappa \in [1, \infty]$ such that $p \leq q$. Let r be a real number such that $r > n/p$, if $p > 1$, or $r \geq n$, if $p = 1$. Finally, let $k = \lceil r \rceil - 1$, $\mathfrak{K} = \dim P_k$, and $l_0 = r - n/p + n/q$. Then, there exists a constant $R > 1$ (dependent on n and r) and, for any $M' \geq 1$, there exists two constants C (dependent on M', n, r, p, q , and \varkappa) and $K \geq 1$ (explicitly dependent on $\lceil l \rceil - l$ and q , cf. (10)), satisfying the following property: for any $d > 0$ and any $t \in \mathbb{R}^n$, the open ball $B(t, Rd)$ contains \mathfrak{K} closed balls $\mathcal{B}_1, \dots, \mathcal{B}_{\mathfrak{K}}$ of radius d such that, for any $v \in W^{r,p}(\overline{B}(t, M'Rd))$, for any $b \in \Pi_{i=1}^{\mathfrak{K}} \mathcal{B}_i$ and $l \in [0, l_{\max}]$,*

$$(11) \quad |v|_{l,q,\overline{B}(t,M'Rd)} \leq C \cdot K \left(d^{r-l-n/p+n/q} |v|_{r,p,\overline{B}(t,M'Rd)} + d^{n/q-l} \|v\|_{\varkappa} \right),$$

where we have let $l_{\max} := \lceil l_0 \rceil - 1$, or $l_{\max} := l_0$ if the following additional conditions hold: $r \in \mathbb{N}^*$ and either (i) $p < q < \infty$ and $l_0 \in \mathbb{N}$, (ii) $(p, q) = (1, \infty)$, or (iii) $1 \leq p = q \leq \infty$.

Proof. The case that $l \in \mathbb{N}$ is established by [1, Proposition 3.4]. Suppose that $l \notin \mathbb{N}$. The hypotheses imply that $l_{\max} \in \mathbb{N}$, so that $\lceil l \rceil \leq l_{\max}$, and thus the result follows by combining (11) for $l = \lceil l \rceil$ with Corollary 2.6, using the fact that $(M'R)^{\lceil l \rceil - l} \leq M'R$, and substituting the dependence on $\lceil l \rceil$ and R with n, r, p and q . \square

2.3. Sobolev Bounds. For $p \leq q$, the following result generalizes [1, Theorem 4.1] to bound fractional order Sobolev semi-norms. No generalization to bound fractional order Sobolev semi-norms is made for $p > q$, since we have not obtained the relation [1, Eq. 2.1] for the case that l is fractional.

Theorem 2.8. *Let $p, q, \varkappa \in [1, \infty]$ and let r be a real number and μ a nonnegative integer such that $r - \mu \geq n$, if $p = 1$, $r - \mu > n/p$ if $1 < p < \infty$, or $r - \mu \in \mathbb{N}^*$ if $p = \infty$. Let $l_0 = r - \mu - n(1/p - 1/q)_+$ and $\gamma = \max\{p, q, \varkappa\}$. Then, there exist three positive constants \mathfrak{d}_r (dependent on θ, p, n, r and μ), C (dependent on Ω, n, r, p, q , and \varkappa), and $K \geq 1$ (explicitly dependent on $\lceil l \rceil - l$ and q , cf. (10)), satisfying the following property: for any set $A \subset \overline{\Omega}$ (or $A \subset \Omega$ if $p = 1$ and $r - \mu = n$) such that $d = \delta(A, \overline{\Omega}) \leq \mathfrak{d}_r$, for any $u \in W^{r,p}(\Omega)$ and, if $p \leq q$ then for any real number $l \in [0, l_{\max}]$, otherwise if $p > q$ then for any integer $l = 0, \dots, l_{\max}$,*

$$(12) \quad |u|_{l,q,\Omega} \leq C \cdot K \left(d^{r-l-n(1/p-1/q)_+} |u|_{r,p,\Omega} + d^{n/\gamma+\mu-l} \left\| \prod_{|\alpha|=\mu} \partial^\alpha u|_A \right\|_{\varkappa} \right),$$

where we have let $l_{\max} := \lceil l_0 \rceil - 1$, or $l_{\max} := l_0$ if the following additional conditions hold: $r \in \mathbb{N}^*$ and either (i) $p < q < \infty$ and $l_0 \in \mathbb{N}$, (ii) $(p, q) = (1, \infty)$, or (iii) $p \geq q$.

Proof. The case that $\mu = 0$ and $l \in \mathbb{N}$ is [1, Theorem 4.1]. The proof of this theorem for $\mu = 0$ and $l \notin \mathbb{N}$ can be obtained by reusing the proof of [1, Theorem 4.1], but applying Proposition 2.7 instead of [1, Proposition 3.4], which allows for l to be of fractional order for $p \leq q$, and introduces the constant K . We now consider the case $\mu > 0$. Let α be a multi-index such that $|\alpha| = \mu$ and therefore $\partial^\alpha u \in W^{r-\mu,p}(\Omega)$.

It follows from the case that $\mu = 0$ that in the situation required by the present hypotheses

$$|\partial^\alpha u|_{l-\mu, q, \Omega} \leq C \cdot K \left(d^{r-l-n(1/p-1/q)+} |\partial^\alpha u|_{r-\mu, p, \Omega} + d^{n/\gamma+\mu-l} \|\partial^\alpha u|_A\|_{\mathcal{X}} \right).$$

We have for all α satisfying $|\alpha| = \mu$ that

$$|\partial^\alpha u|_{r-\mu, p, \Omega} \leq |u|_{r, p, \Omega}$$

and

$$\|\partial^\alpha u|_A\|_{\mathcal{X}} \leq \left\| \prod_{|\beta|=\mu} \partial^\beta u|_A \right\|_{\mathcal{X}}.$$

The result follows immediately for $q = \infty$. Otherwise, if $1 \leq q < \infty$, using that

$$|u|_{l, q, \Omega}^q \leq \sum_{|\alpha|=\mu} |\partial^\alpha u|_{l-\mu, q, \Omega}^q$$

the results follows with an additional factor $(\#\{\alpha : |\alpha| = \mu\})^{1/q}$ in the constant C , whose dependence on μ can be substituted with n, r and p . \square

Corollary 2.9. *Given the situation of Theorem 2.8 with a constant \mathfrak{d}_r now dependent on θ, ρ, n, r, p and q , and the additional assumption that $r - l \in \mathbb{N}$, then we have*

$$\|u\|_{l, q, \Omega} \leq C \cdot K \left(d^{r-l-n/p+n/q} \|u\|_{r, p, \Omega} + d^{n/\gamma+\mu-l} \left\| \prod_{|\alpha| \leq \mu} \partial^\alpha u|_A \right\|_{\mathcal{X}} \right).$$

Proof. From Theorem 2.8, there exists three positive constants $\mathfrak{d}_r(\theta, \rho, n, r, p, q)$, $C(\Omega, n, r, p, q, \mathcal{X})$, and $K([l] - l, q) \geq 1$, cf. (10), such that for $d \leq \mathfrak{d}_r$ and $\eta = 0, \dots, [l]$

$$\begin{aligned} |u|_{\eta, q, \Omega} &\leq C \cdot K \left(d^{r-l-n/p+n/q} |u|_{r-l+\eta, p, \Omega} \right. \\ &\quad \left. + d^{n/\gamma+(\eta+\mu-[l])_+-\eta} \left\| \prod_{|\alpha|=(\eta+\mu-[l])_+} \partial^\alpha u|_A \right\|_{\mathcal{X}} \right) \end{aligned}$$

and for $\eta = l$ that

$$|u|_{l, q, \Omega} \leq C \cdot K \left(d^{r-l-n/p+n/q} |u|_{r, p, \Omega} + d^{n/\gamma+\mu-l} \left\| \prod_{|\alpha|=\mu} \partial^\alpha u|_A \right\|_{\mathcal{X}} \right).$$

We have taken the constants to be the minimum or maximum over $\eta = 0, \dots, [l]$, l as required. Additionally we have restricted \mathfrak{d}_r to be at most one. For $\eta = 0, \dots, [l]$, we have applied Theorem 2.8 with $r = r - l + \eta$ and $\mu = (\eta + \mu - [l])_+$, introducing a dependence of \mathfrak{d}_r on p and q through l , which along with n and r has substituted for the dependence on μ . It also follows for all $\eta = 0, \dots, [l]$ that

$$(13) \quad \left\| \prod_{|\alpha|=(\eta+\mu-[l])_+} \partial^\alpha u|_A \right\|_{\mathcal{X}} \leq \left\| \prod_{|\alpha| \leq \mu} \partial^\alpha u|_A \right\|_{\mathcal{X}},$$

with a similar bound holding for $\left\| \prod_{|\alpha|=\mu} \partial^\alpha u|_A \right\|_{\varkappa}$. It follows from

$$(\eta + \mu - [l])_+ - \eta \geq \mu - [l] \geq \mu - l$$

and $\mathfrak{d}_r \leq 1$ that for all $d \leq \mathfrak{d}_r$,

$$(14) \quad d^{n/\gamma + (\eta + \mu - [l])_+ - \eta} \leq d^{n/\gamma + \mu - l}.$$

It follows from $r - l \in \mathbb{N}$ that for each $\eta = 0, \dots, [l]$ we have $r - l + \eta \in \mathbb{N}$ and $0 \leq r - l + \eta \leq r - l$, which implies that

$$(15) \quad |u|_{r-l+\eta, p, \Omega} \leq \|u\|_{r, p, \Omega}.$$

Combining the preceding bounds we obtain for all $d \leq \mathfrak{d}_r$ and $\eta = 0, \dots, [l]$, l that

$$(16) \quad |u|_{\eta, q, \Omega} \leq C \cdot K \left(d^{r-l-n/p+n/q} \|u\|_{r, p, \Omega} + d^{n/\gamma + \mu - l} \left\| \prod_{|\alpha| \leq \mu} \partial^\alpha u|_A \right\|_{\varkappa} \right).$$

If $q = \infty$ then the result follows immediately. If $1 \leq q < \infty$ it follows from (16) that

$$\begin{aligned} \|u\|_{l, q, \Omega} &\leq C \cdot K \cdot ([l] + 2)^{1/q} \left(d^{r-l-n/p+n/q} \|u\|_{r, p, \Omega} \right. \\ &\quad \left. + d^{n/\gamma + \mu - l} \left\| \prod_{|\alpha| \leq \mu} \partial^\alpha u|_A \right\|_{\varkappa} \right). \end{aligned}$$

The result then follows by incorporating the constant $([l] + 2)^{1/q}$ into C , with the dependence on $[l]$ substituted with n, r, p and q . \square

Only the case that $p = q = \varkappa = 2$ will be used in Section 3.

Corollary 2.10. *Let r be a real number, μ be a nonnegative integer such that $r - \mu > n/2$. Then, there exist three positive constants \mathfrak{d}_r (dependent on θ, ρ, n and r), C (dependent on Ω, n and r), and K (explicitly dependent on $[l] - l$, cf. (10) with $q = 2$) satisfying the following property: for any set $A \subset \overline{\Omega}$, such that $d = \delta(A, \overline{\Omega}) \leq \mathfrak{d}_r$, $u \in W^{r, 2}(\Omega)$ and real number $l \in [0, r - \mu]$ such that $r - l \in \mathbb{N}$,*

$$(17) \quad \|u\|_{l, 2, \Omega} \leq C \cdot K \left(d^{r-l} \|u\|_{r, 2, \Omega} + d^{n/2 + \mu - l} \left\| \prod_{|\alpha| \leq \mu} \partial^\alpha u|_A \right\|_2 \right).$$

3. APPLICATION: UNSYMMETRIC MESHLESS METHODS FOR OPERATOR EQUATIONS

In this section, only the dependence of constants on the discretization parameters r and s is explicitly stated since we regard the spaces and mappings involved as fixed.

We now apply the sampling inequality stated in Corollary 2.10 to Schaback's framework for unsymmetric meshless methods for operator equations [12]. Due to unaddressed issues contained in its original formulation, we use a modified version stated in this section. On a technical level, it differs substantially, e.g., certain spaces have been eliminated, the inequalities apply over possibly different spaces, and the proof of the error bound has been slightly modified, but the underlying ideas are the same and entirely due to Schaback [12].

The framework provides an error bound for meshless methods which approximately solve a linear operator equation in the following setting. The first requirement is a continuous and bijective linear operator $L : U \rightarrow F$ mapping from the *solution space* to the *data space*. The spaces U and F are assumed to be complete in order to ensure the boundedness of $L^{-1} : F \rightarrow U$. It is also assumed that the exact solution $u^* \in \tilde{U}$ where $\tilde{U} \subset U$ is called the *regularity subspace*. We denote $\tilde{F} := L\tilde{U}$. The framework requires a scale of finite-dimensional trial subspaces $U_r \subset \tilde{U}$ equipped with a projector $\Pi_r : \tilde{U} \rightarrow U_r$. The framework requires a linear, continuous, and bijective *test mapping* $\Lambda : F \rightarrow T$, where the *test space* T is assumed to be complete in order to ensure the boundedness of Λ^{-1} . We denote $\tilde{T} := \Lambda\tilde{F}$. Test data from T is discretized into finite-dimensional test subspaces T_s with a *test discretization* mapping

$$(18) \quad \pi_s : T \rightarrow T_s$$

the operator norm of which must be bounded by a constant, which is independent of s . It follows that the operator norm of

$$\pi_s \Lambda L : U \rightarrow T_s$$

is bounded similarly since

$$\|\pi_s \Lambda L\|_{U \rightarrow T_s} \leq \|\pi_s\|_{T \rightarrow T_s} \|\Lambda L\|_{U \rightarrow T}$$

In order to apply the error bound of Schaback's framework a number of inequalities must be supplied. The first of these is the *trial space approximation property*

$$(19) \quad \|u - \Pi_r u\|_U \leq \epsilon(r) \|u\|_{\tilde{U}} \text{ for all } u \in \tilde{U}.$$

The second inequality is the test discretization's *stability condition*

$$(20) \quad \|\Lambda L u_r\|_T \leq \beta(s) \|\pi_s \Lambda L u_r\|_{T_s} \text{ for all } u_r \in U_r.$$

If the *stability factor* $\beta(s)$ grows as the test discretization is refined, i.e., as s decreases towards zero, then the order of convergence in the final error bound (23) will be less than that provided by the trial space approximation property (19). When the stability factor does not grow, the test discretization is called *uniformly stable*. The final inequality required by Schaback's framework involves a numerical method capable of providing an approximate solution $u_{r,s}^* \in U_r$ which satisfies the *numerical method approximation property*

$$(21) \quad \|\pi_s \Lambda (L u_{r,s}^* - f)\|_{T_s} \leq C \|\pi_s \Lambda L\|_{U \rightarrow T_s} \epsilon(r) \|u^*\|_{\tilde{U}}.$$

In particular, if the numerical method computes $u_{r,s}^* \in U_r$ which minimizes the left hand side of (21) then the constant is at most one, since

$$(22) \quad \begin{aligned} \|\pi_s \Lambda (L u_{r,s}^* - f)\|_{T_s} &\leq \|\pi_s \Lambda L (\Pi_r u^* - u^*)\|_{T_s} \\ &\leq \|\pi_s \Lambda L\|_{U \rightarrow T_s} \|\Pi_r u^* - u^*\|_U \\ &\leq \|\pi_s \Lambda L\|_{U \rightarrow T_s} \epsilon(r) \|u^*\|_{\tilde{U}}. \end{aligned}$$

Theorem 3.1. [12, Theorem 1] *Given the setting stated above, if the inequalities (19), (20), and (21) are satisfied then the following error bound holds:*

$$\|u^* - u_{r,s}^*\|_U \leq \left(1 + \beta(s) \left\|(\Lambda L)^{-1}\right\|_{T \rightarrow U} \|\pi_s \Lambda L\|_{U \rightarrow T_s} (1 + C)\right) \epsilon(r) \|u^*\|_{\tilde{U}}.$$

Proof. We have that

$$\begin{aligned} \|u^* - u_{r,s}^*\|_U &\leq \|u^* - \Pi_r u^*\|_U + \|\Pi_r u^* - u_{r,s}^*\|_U \\ &\leq \epsilon(r) \|u^*\|_{\tilde{U}} + \|\Pi_r u^* - u_{r,s}^*\|_U \end{aligned}$$

$$\begin{aligned} \|\Pi_r u^* - u_{r,s}^*\|_U &\leq \|(\Lambda L)^{-1}\|_{T \rightarrow U} \|\Lambda L (\Pi_r u^* - u_{r,s}^*)\|_T \\ &\leq \beta(s) \|(\Lambda L)^{-1}\|_{T \rightarrow U} \|\pi_s \Lambda L (\Pi_r u^* - u_{r,s}^*)\|_{T_s} \\ &\leq \beta(s) \|(\Lambda L)^{-1}\|_{T \rightarrow U} (\|\pi_s \Lambda L (\Pi_r u^* - u^*)\|_{T_s} \\ &\quad + \|\pi_s \Lambda L (u^* - u_{r,s}^*)\|_{T_s}) \\ &= \beta(s) \|(\Lambda L)^{-1}\|_{T \rightarrow U} \|\pi_s \Lambda L\|_{U \rightarrow T_s} \epsilon(r) \|u^*\|_{\tilde{U}} (1 + C) \end{aligned}$$

$$\begin{aligned} \|u^* - u_{r,s}^*\|_U &\leq \epsilon(r) \|u^*\|_{\tilde{U}} \\ &\quad + \beta(s) \|(\Lambda L)^{-1}\|_{T \rightarrow U} \|\pi_s \Lambda L\|_{U \rightarrow T_s} \epsilon(r) \|u^*\|_{\tilde{U}} (1 + C). \end{aligned}$$

□

The stability condition (20) can be established using an *inverse estimate*

$$(23) \quad \|u_r\|_{\tilde{U}} \leq \gamma(r) \|u_r\|_U \text{ for all } u_r \in U_r,$$

a *sampling inequality*

$$(24) \quad \|f\|_T \leq C (\alpha(s) \|f\|_{\tilde{T}} + \beta(s) \|\pi_s f\|_{T_s}) \text{ for all } f \in \tilde{T},$$

and ensuring that a *fine enough test discretization* is chosen such that

$$(25) \quad C \alpha(s) \gamma(r) \|\Lambda L\|_{\tilde{U} \rightarrow \tilde{T}} \|(\Lambda L)^{-1}\|_{T \rightarrow U} \leq \frac{1}{2},$$

where C is the constant appearing in (24). Typically, $\gamma(r) \rightarrow \infty$ as $r \rightarrow 0$, while $\alpha(s) \rightarrow 0$ as $s \rightarrow 0$.

Proposition 3.2. [12, Theorem 2] *If (23), (24), and (25) hold then so does (20).*

Proof. We have that

$$\begin{aligned} \|\Lambda L u_r\|_T &\leq C (\alpha(s) \|\Lambda L u_r\|_{\tilde{T}} + \beta(s) \|\pi_s \Lambda L u_r\|_{T_s}) \\ &\leq C (\alpha(s) \|\Lambda L\|_{\tilde{U} \rightarrow \tilde{T}} \|u_r\|_{\tilde{U}} + \beta(s) \|\pi_s \Lambda L u_r\|_{T_s}) \\ &\leq C (\alpha(s) \|\Lambda L\|_{\tilde{U} \rightarrow \tilde{T}} \gamma(r) \|u_r\|_U + \beta(s) \|\pi_s \Lambda L u_r\|_{T_s}) \\ &\leq C (\alpha(s) \|\Lambda L\|_{\tilde{U} \rightarrow \tilde{T}} \gamma(r) \|(\Lambda L)^{-1}\|_{T \rightarrow U} \|\Lambda L u_r\|_T \\ &\quad + \beta(s) \|\pi_s \Lambda L u_r\|_{T_s}) \\ &\leq \frac{1}{2} \|\Lambda L u_r\|_T + C \beta(s) \|\pi_s \Lambda L u_r\|_{T_s} \end{aligned}$$

and the result follows by incorporating the constant $2C$ in $\beta(s)$. □

3.1. Convergence Results for the Poisson Problem. We consider the example from [12, Section 4.1], a Poisson problem with mixed, inhomogeneous boundary data: let Ω be a bounded domain in \mathbb{R}^d with a Lipschitz-continuous boundary. We denote $\Omega_1 := \Omega$, $\Omega_2 := \Gamma^D \subset \partial\Omega$, and $\Omega_3 = \Gamma^N \subset \partial\Omega$ so that the dimension of each domain is given by $n_1 = n$, and $n_2, n_3 = n - 1$. Let m, \tilde{m} be nonnegative real numbers such that $\tilde{m} - m \in \mathbb{N}$, and

$$\begin{aligned}
 (m_1, m_2, m_3) &:= (m, m + 3/2, m + 1/2) \\
 U &:= H^{m+2}(\Omega) \\
 F &:= F^1 \times F^2 \times F^3 \\
 &:= H^{m_1}(\Omega_1) \times H^{m_2}(\Omega_2) \times H^{m_3}(\Omega_3) \\
 (26) \quad Lu &:= \left(-\Delta u, u|_{\Gamma^D}, \frac{\partial u}{\partial n}|_{\Gamma^N} \right),
 \end{aligned}$$

with analogous definitions made for $(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3)$, \tilde{U} , and \tilde{F} . With the space F equipped with the norm $\|\cdot\|_F^2 := \|\cdot\|_{F^1}^2 + \|\cdot\|_{F^2}^2 + \|\cdot\|_{F^3}^2$, it follows that the linear operator L , as defined above, is continuously invertible either as $L : U \rightarrow F$ or $L : \tilde{U} \rightarrow \tilde{F}$.

We assume that the solution comes from \tilde{U} and that the trial space U_r is chosen such that the trial space approximation property (19) holds with $\epsilon(r) = O(r^{\tilde{m}-m})$, a property satisfied by kernel-based meshless trial spaces, c.f. Narcowich et al. [7, 8], and finite-element trial spaces [3, Theorem 4.5.11]. We also assume that the inverse estimate (23) holds with $\gamma(r) = O(r^{m-\tilde{m}})$, as is the case for finite-element trial spaces [3, Theorem 4.4.20]. Obtaining an inverse estimate with the expected factor $\gamma(r) = O(r^{m-\tilde{m}})$ appears to be an open problem for kernel-based meshless trial spaces. Narcowich et al. [8] provide an inverse estimate with the expected factor for the case of Sobolev spaces over \mathbb{R}^n . Both Schaback and Wendland [11], and Duan [4] provide inverse estimates for Sobolev spaces over a domain. Unfortunately, the factor involved in these inverse estimates are worse than the finite-element case. Further progress on this problem is expected to be reported in the thesis of Rieger [9].

We consider the case of strong testing here, which means that the test mapping $\Lambda : F \rightarrow T$ is just the identity mapping and that each test space T^k coincides with the corresponding data space F^k . Weak testing is also possible, in which case the test functionals integrate functions in F^k against test functions, resulting in the test data in each T^k acquiring additional smoothness. This is discussed in detail by Schaback [12, 14]. Each domain is discretized onto finite subsets $Y_s^k \subset \Omega_k$, with the same fill distance $s = \delta(Y_s^k, \Omega_k)$. Furthermore, they are assumed to satisfy the property that $\#Y_s^k$ is bounded by s^{-n_k} up to a constant, as is the case for domain discretization with a uniformly bounded mesh ratio [12]. We define discrete test spaces

$$T_s^k := \mathbb{R}^{\#\{\alpha : |\alpha| \leq \mu_k\} \cdot \#Y_s^k}$$

equipped with a norm

$$(27) \quad \|\cdot\|_{T_s^k} := s^{n_k/2} \|\cdot\|_2$$

and a test discretization $\pi_s^k : T^k \rightarrow T_s^k$

$$(28) \quad \pi_s^k f_k := \prod_{|\alpha| \leq \mu_k} \partial^\alpha f_k|_{Y_s^k} \text{ for all } f_k \in T^k$$

where μ_k is an integer such that $m_k - \mu_k - n_k/2 > 0$, and furthermore this difference is independent of k . The discrete test space $T_s := T_s^1 \times T_s^2 \times T_s^3$ is defined and equipped with a norm, analogously to F and T . The test space T is then equipped with a test discretization $\pi_s : T \rightarrow T_s$ defined by

$$(29) \quad \pi_s f := (\pi_s^1 f_1, \pi_s^2 f_2, \pi_s^3 f_3) \text{ for all } f = (f_1, f_2, f_3) \in T,$$

Proposition 3.3. *If for each k , $m_k - \mu_k - n_k/2 > 0$ then $\pi_s : T \rightarrow T_s$ is well-defined and the operator norm $\|\pi_s\|_{T \rightarrow T_s}$ is bounded independently of s .*

Proof. Suppose $f = (f_1, f_2, f_3) \in T$. Since $m_k - \mu_k > n_k/2$ we have from the Sobolev embedding theorem that $T^k \hookrightarrow C^{\mu_k}(\overline{\Omega_k})$ and therefore the test discretization is both well-defined and there exists some constant independent of $f = (f_1, f_2, f_3)$ such that for each f_k ,

$$\|f_k\|_{C^{\mu_k}(\overline{\Omega_k})} \leq C \|f_k\|_{T^k}.$$

Since $\#Y_s^k$ is bounded by s^{-n_k} up to some constant which is independent of s , it follows that

$$\begin{aligned} \|\pi_s f\|_{T_s}^2 &= \sum_{k=1}^3 \|\pi_s^k f_k\|_{T_s^k}^2 \\ &= \sum_{k=1}^3 s^{n_k} \sum_{x \in Y_s^k} \sum_{|\alpha| \leq \mu_k} |\partial^\alpha f(x)|^2 \\ &\leq \sum_{k=1}^3 s^{n_k} \|f\|_{C^{\mu_k}(\overline{\Omega_k})}^2 \# \{\alpha : |\alpha| \leq \mu_k\} \cdot \#Y_s^k \\ &\leq C \sum_{k=1}^3 \|f\|_{T^k}^2 = C \|f\|_T^2. \end{aligned}$$

□

Proposition 3.4. *There exists a constant s_0 such that for all $s \leq s_0$ a sampling inequality (24) holds with a constant C for the test space T and test discretization $\pi_s : T \rightarrow T_s$ with $\alpha(s) := s^{\tilde{m}-m}$, and $\beta(s) := s^{\mu_1-m_1} = s^{\mu_2-1/2-m_2} = s^{\mu_3-1/2-m_3}$.*

Proof. From Corollary 2.10 and (27), it follows that for each k there exist constants C_k and s_k such that for $s \leq s_0 := \min(1, s_1, s_2, s_3)$,

$$\begin{aligned} \|f_k\|_{T^k} &\leq C_k \left(\alpha(s) \|f_k\|_{\tilde{T}^k} + s^{\mu_k-m_k} \|\pi_s^k f_k\|_{T_s^k} \right) \\ &\leq C_k \left(\alpha(s) \|f_k\|_{\tilde{T}^k} + s^{\mu_1-m_1} \|\pi_s^k f_k\|_{T_s^k} \right) \end{aligned}$$

since $\mu_1 - m_1 \leq \mu_2 - m_2 = \mu_3 - m_3$. The result then follows with a constant C by combining the preceding inequalities. □

We assume that the s is sufficiently small to satisfy the requirements of Proposition 3.4 and (25). Even in the fractional case, the sampling inequality introduced here provides $\alpha(s) = s^{\tilde{m}-m}$ which shrinks as rapidly as the expected inverse estimate factor $\gamma(r) = r^{m-\tilde{m}}$ grows and thus s and r can be kept proportional. This is in contrast to previous sampling inequalities which necessarily introduce a factor $\alpha(s) = s^{\tilde{m}-\lceil m \rceil}$ when bounding fractional order Sobolev norms, requiring the test discretization to be refined more rapidly than the trial discretization and thus

$U = H^{m+2}(\Omega) =$	$H^0(\Omega)$	$H^4(\Omega)$	$H^5(\Omega)$	$H^6(\Omega)$
$\mu_1 = 0$	None	$\tilde{m} - m - 2$	$\tilde{m} - m - 3$	$\tilde{m} - m - 4$
$\mu_1 = 1$	None	None	$\tilde{m} - m - 2$	$\tilde{m} - m - 3$
$\mu_1 = 2$	None	None	None	$\tilde{m} - m - 2$
$\mu_1 = 3$	None	None	None	None

TABLE 1. Order of convergence in various Sobolev norms established by a modified formulation of Schaback's framework, using trial spaces with optimal properties and strong testing with various order test discretizations to solve two- or three-dimensional Poisson problems.

diminishing the order of convergence by $\lceil m \rceil - m$. If the function $u_{r,s}^* \in U_r$ which minimizes the left hand side of (21) has been computed, then Schaback's framework provides the error bound (23) with constant $C = 1$. The order of convergence established by this error bound, in terms of both the trial and test discretization, is then given by $\beta(h)\epsilon(h)$ and satisfies

$$\beta(h)\epsilon(h) = O\left(h^{(\tilde{m}-m)+(\mu_1-m_1)}\right).$$

Table 1 states particular convergence results using various Sobolev norms and test discretizations in two- and three-dimensions. These results show that, for convergence in higher order norms, the highest order of convergence is obtained using a higher order test discretization introduced here.

We note that to obtain a uniformly stable test discretization with $\beta(s) = 1$ would require choosing the order μ_k of each test discretization to be equal to that of the order m_k of the test space. Unfortunately, this does not seem to be possible: in order for the test discretizations operator norm to be bounded independently of s , the order of each test space T^k is required to be greater than that of the test discretization μ_k by at least $n_k/2$. It follows that the order of convergence, in terms of both the trial and test discretization, provided by this modified formulation of Schaback's framework is always less than that of the trial space approximation property. Another consequence is that the order of U must be at least $2 + n/2$, and therefore convergence in the L^2 norm can only be concluded suboptimally from convergence results in higher order Sobolev norms, using strong testing in this modified formulation of Schaback's framework.

4. CONCLUSIONS

We have further generalized the sampling inequalities of Arcangéli et al. [1], Madych [6], and Wendland and Rieger [15], to optimally bound fractional order Sobolev semi-norms, and to incorporate higher order data into the bound. When used in a modified formulation of Schaback's framework to prove convergence rates for unsymmetric meshless methods this new sampling inequality has two benefits:

- (1) It results in more optimal estimates for problems involving fractional order Sobolev spaces, particularly by providing a more optimal constant $\alpha(s)$.
- (2) For convergence in higher order Sobolev norms, higher order results are obtained using a higher order test discretization in comparison to the zero order test discretization.

The zero order test discretization has been widely employed in practice, and corresponds to what is usually called Kansa's method or unsymmetric collocation. On the other hand higher order testing has not, and its value in practical applications requiring convergence in stronger norms is an open question worthy of further study.

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